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Surf and run-up on a beach: a uniform bore

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A numerical solution is obtained to describe the behaviour of a uniform bore over a sloping beach and the subsequent run-up and back-wash. The results exhibit features which have only previously been described in a qualitative manner. These include the formation of a landward-facing bore in the back-wash. A comprehensive set of results are presented for a typical initial subcritical bore height ratio.

1. Introduction

This paper is the first result of a continuing project to develop and examine theoretical solutions for wave motions on a beach. A number of recent books (Meyer 1972; Radok & Provis 1977; Hails & Carr 1975) describe various aspects of the situations that occur when water waves are incident on beaches. The particular problem considered in this work is the motion near the shoreline. In most circumstances waves break as they approach the shore, and in their final approach to the shoreline form a turbulent bore. Unbroken waves reach the shoreline under three main conditions: (i) where the beach is steep; (ii) where the incident waves are of very gentle slope and (iii) where there is so much relatively shallow water in front of the shore that bottom friction dissipates most of the energy in the incoming wave. Waves break in various ways, but in most circumstances there is a region near the shoreline where the waves have short steep turbulent fronts, that is bores, and otherwise have very gentle slopes. This region which we call the 'bore region' can be large or small depending on the slope of the beach and the incident waves.

Appropriate approximate equations to describe the wave dynamics within the bore region are the finite-amplitude shallow-water equations with the steep wave fronts represented by bores – mathematically by discontinuities. On some beaches broken waves soon become bores and a substantial region may be described by this mathematical model. At the very least this model applies to the run-up, even on a very steep beach. It is of particular interest to study the nearshore region since the highest water levels and greatest sediment transport tend to be at the shoreline.

This work does not include any direct representation of dissipative effects; dissipation is implicit in the bore representation but the basic flow is inviscid. As a start in developing solutions we consider this important omission to have advantages. There are a number of indications that the commonly used Chézy friction terms are unsuited to unsteady flow problems. Such terms are, after all, empirical terms developed from steady river flows. Analytic solutions are possible if no friction is included and this gives an important check on the analysis. In this paper we describe

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an accurate method of solving the equations and include comparisons with existing analytic results. Most of the analytic results are well described and discussed by Meyer & Taylor (1972). Further advantages of neglecting friction, at this stage, are that the value of the beach slope, γ, can be scaled from the equations and comparisons with experiment can show more directly the effect of dissipation. One disadvantage is that the numerical treatment of the shoreline has not been as straightforward as one would wish.

The 'uniform bore' described in the title is an idealized problem. It has the advantage that it includes most of the features that can occur in the bore region and yet all motion near the shore ceases after a finite time. The initial condition is still water over a beach of uniform slope. A boundary condition must be specified at the seaward boundary which is taken as the edge of the sloping region. We apply boundary conditions corresponding to a bore of given strength approaching along a region of uniform depth with a uniform flow behind it. Since the boundary conditions must be specified in terms of variables on incoming characteristics this does *not* necessarily represent a uniform flow at the seaward boundary.

In the ensuing motion the bore advances across the still water to the shoreline and causes the water to run up the beach. Eventually the instantaneous shoreline reaches a maximum run-up height and the water flows back down the beach. The flow soon becomes supercritical and the shoreline would recede continually if it was not brought very nearly to rest by the formation of a shoreward-facing bore in the interior of the flow. After further gentle oscillations all wave motion propagates out of the integration region through the seaward boundary leaving still water at a higher level. This higher level is readily calculated since it corresponds to the level that would be obtained by reflexion of the incoming characteristics at a vertical wall. In the case where flow behind the bore is initially subcritical a reflected bore would form in the uniform depth region outside the seaward boundary and affect the motion in the shoreline region by changing the incoming characteristics. This is not considered here and the simpler case of 'constant' boundary conditions is imposed.

Different solutions are obtained by specifying varying seaward boundary conditions. Realistic simulations of waves on a beach can be obtained by using boundary conditions which correspond to water motions shoreward of the region in which waves break. Further papers are in preparation describing such solutions and comparisons with experiments.

The equations used to approximate the flow are given in the next section. Their characteristics and analytic representation of incident waves are discussed. Also included is a local analysis of shoreline motion and a brief summary of analytic solutions. Section 3 describes the numerical method of solution. The main body of the integration is performed using an explicit Lax–Wendroff scheme as given in Richtmyer & Morton (1967). Most attention in developing the model has to be paid to the motion of the shoreline point and so a numerical method known to be straightforward was considered desirable for the bulk of the flow. It is also particularly important that such a method allows bores to form within the flow interior.

Results of numerical computation are discussed in §§ 4 and 5. The former section gives comparisons with analytical solutions. Good agreement is found except in regions where the discretization gives insufficient resolution. In any practical case, the finite width of a turbulent bore precludes exact comparison with analysis or computation. The results of computations for a typical subcritical uniform bore are presented in detail in § 6. Experiments performed by Miller (1968) appear to be the only published experimental results for a uniform bore. A comparison is made with the computations in the concluding section.

2. Mathematical model

The finite-amplitude shallow-water equations are a good approximation to inviscid flow when any variations of surface level or mean velocity have a length-scale many times the maximum depth of water (e.g. see Peregrine 1972, for a derivation). Such an approximation comes from assuming both vorticity and vertical accelerations are negligibly small. Assuming a beach of uniform slope γ, the governing equations for mass and momentum may be written:

$$\frac{\partial h^*}{\partial t^*}+\frac{\partial(h^*u^*)}{\partial x^*} = 0, \tag{1}$$

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+g\cos\gamma\frac{\partial h^*}{\partial x^*}+g\sin\gamma = 0, \tag{2}$$

where $h^*(x^*, t^*)$ is the total depth of water and $u^*(x^*, t^*)$ is the depth-averaged water velocity. The co-ordinate parallel to the bottom is given by x^* and t^* refers to the time.

Dimensionless variables are chosen to eliminate the beach slope from the equations. These are

$$\left.\begin{aligned} h &= (\cos\gamma)\, h^*/h_1, & u &= u^*(gh_1)^{-\frac{1}{2}},\\ x &= (\sin\gamma)\, x^*/h_1, & t &= (\sin\gamma)\, t^*(g/h_1)^{\frac{1}{2}}, \end{aligned}\right\} \tag{3}$$

where h_1 is a chosen reference depth, e.g. the depth of water at the chosen seaward boundary. The equations then become

$$h_t+(hu)_x = 0, \tag{4}$$

$$u_t+uu_x+h_x+1 = 0. \tag{5}$$

We note that there is no need for γ to be small. Even on steep beaches where no bore forms the thin layer of run-up can be described by these equations. However, in this particular work we have assumed that γ is sufficiently small that $1-\cos\gamma$ is negligible; there is in this case no need to distinguish between distance measured along the beach and in the horizontal.

Equations (4) and (5) can be written in several equivalent forms; the usefulness of each depending on the particular approach to the problem. A form useful for analysis is obtained on defining characteristic variables (Riemann invariants) α, β as

$$\alpha = u+2c+t-u_0 \tag{6}$$

and

$$-\beta = u-2c+t-u_0, \tag{7}$$

where c is the local long-wave velocity defined by $c = h^{\frac{1}{2}}$ and u_0 is any constant of integration. The equations (4) and (5) can then be replaced by the four relations

$$x_\beta = (u+c)t_\beta, \quad u_\beta+2c_\beta+t_\beta = 0; \tag{8}$$

$$x_\alpha = (u-c)t_\alpha, \quad u_\alpha-2c_\alpha+t_\alpha = 0. \tag{9}$$

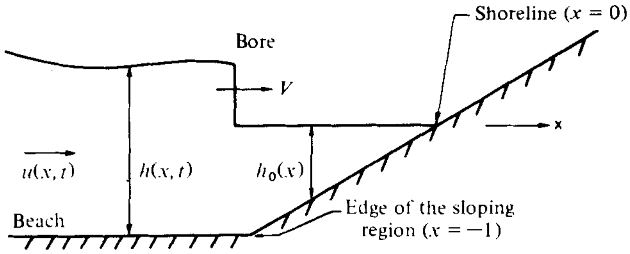

FIGURE 1. Definition sketch.

Variables x, t, u and c are considered as dependent variables with α and β as the independent variables. Relations (8) define paths α = constant along lines in the (x, t) plane given by $dx = (u+c)dt$ whilst (9) defines paths β = constant along intersecting lines $dx = (u-c)dt$. Such paths in space–time define the advancing and receding characteristics respectively and correspond physically to paths of infinitesimal wave disturbances.

Presence of a bore within the problem means that the shallow-water equations need to be supplemented by bore relations in the form of internal boundary conditions across a moving discontinuity of velocity and water height. A discussion of relations valid across a bore is given by Coulson & Jeffrey (1977) in terms of the physical variables, or by Peregrine (1974) referring to the characteristic variables.

The problem of a uniform bore incident onto a sloping beach is envisaged as shown in figure 1. With the variables scaled appropriately the region of sloping beach is given by $x > -1$ with the undisturbed water depth unity at the boundary of the sloping region. The region $x < -1$ comprises of a horizontal bed with an incident uniform bore of height $h_1 = c_1^2$ and a water velocity behind the bore of u_1. Slightly different shallow-water equations apply in the two regions with a corresponding change in the definition of the characteristic variables. For an incident uniform bore we have two cases to consider, viz. subcritical bore ($u_1 < c_1$) and supercritical bore ($u_1 > c_1$). In this paper only the subcritical case is studied in detail. Only the area of the sloping beach is considered in the computations and so we need to interpret the required boundary condition to be specified on the fixed line $x = -1$. The offshore travel of the uniform bore is easily described with the use of the bore relations as all motion is steady. Computation of the solution in the sloping region will start only from time $t = 0$ when the bore first meets the seaward boundary of the region of calculation, that is $x = -1$.

The shallow-water equations (4) and (5) are modified for water of constant depth by dropping the last term (+1) in equation (5). When written in characteristic form they yield the relations $\alpha' = u+2c$ = constant along advancing characteristic lines $dx/dt = u+c$ and $-\beta' = u-2c$ = constant along receding characteristic lines $dx/dt = u-c$. Referring to figure 2 the receding characteristic C emanating into the horizontal region from the edge $x = -1$ at time zero bounds a uniform region I in which both α' and β' are everywhere constant. The advancing characteristics entering the region II originate from I and it thus follows that region II is a simple wave in which the reflected wave forms. Region III is bounded by the bore and the edge of the sloping beach and is provided with boundary data from II in the form of

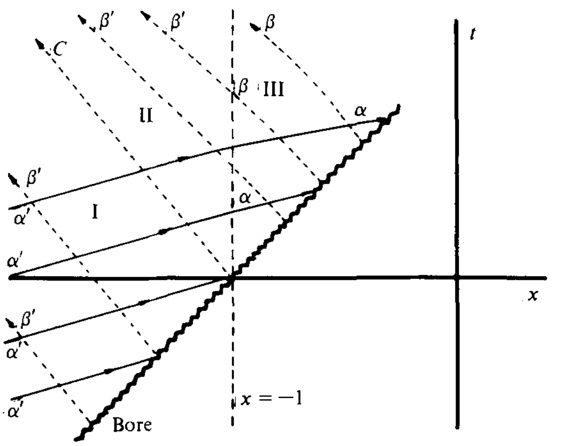

FIGURE 2. Configuration of characteristics for an initially uniform subcritical bore on meeting a sloping region.

values of the Riemann invariant α on the advancing characteristics originating from this boundary. Within II we have

$$\alpha' = u+2c = \alpha_0 = u_1+2c_1, \tag{10}$$

the value α_0 being constant everywhere. Values of the Riemann invariant change at the boundary since we impose continuity of water height and velocity across the boundary. Thus the boundary value of the Riemann invariant along the seaward boundary is given by

$$\alpha = u+2c+t = \alpha_0+t. \tag{11}$$

We notice that (11) does not specify wave variables entirely at the seaward boundary to the sloping region but simply provides a relationship between them. Determination of variables at the seaward boundary must be obtained by combining the values given on the advancing characteristics with the values of the characteristic variables of the opposite family which originate from the bore. In order to avoid overdetermination of our problem only the boundary condition (11) (or its equivalent) is taken. The influence of the reflected bore is not taken into account.

A supercritical bore incident upon a sloping region provides a different seaward boundary condition since the first receding characteristic emanating from the bore at the instant of arrival to the sloping region has positive slope and so extends shorewards into this region. In particular the characteristic line is shorewards of the seaward boundary $x = -1$ and thus values of water height and velocity are the same as those recorded in the uniform region. In this case both values of the dependent variables are almost trivally determined explicitly but the subsequent motion differs from the subcritical case. As water velocity at the seaward boundary is unchanged with increasing time a mass of water is continually supplied to the sloping beach region. This continues irrespective of what happens to the shoreline until such a time that a reflected seaward-facing bore is formed within the interior and travels through the seaward boundary, bringing the water velocity there to subcritical values. Formation of an analogous system of reflected shock waves is recorded by Friedmann (1960) where he calculates the formation of a shock wave within a converging duct for supercritical flow the reflected shock forms upstream of the converging section for subcritical flow. This behaviour for the supercritical case is not as different from the subcritical case as appears at first sight. The receding characteristics from the subcritical bore meet beyond the seaward boundary to form a reflected bore, over the horizontal section. Eventually reflected bores arise in either case and their eventual magnitude is equal to that of a bore reflected from a vertical wall. There is further discussion of bore formation in Hibberd (1979).

The boundary conditions at the shoreline are, quite simply, the depth is zero and the velocity of the shoreline equals that of the water. However, numerical implementation of these conditions proved to be troublesome.

3. Analytic solutions

Meyer & Taylor (1972) review analytic solutions of the shallow-water equations over a beach. More recent papers are those by Spielvogel (1976) and Kajiura (1976). The analytic work falls naturally into two parts: smooth solutions or solutions with bores. The solutions without bores represent waves which are fully reflected since the equations used are non-dissipative; for example the standing-wave solutions of Carrier & Greenspan (1958). These and some solutions of Spielvogel (1976), who considered an initial waveform with water at rest, have been used to check the numerical solution.

The more important solutions for bores are those of Keller, Levine & Whitham (1960) and of Meyer and his associates (summarized in Meyer & Taylor, 1972). These show that the bore 'collapses' to have zero height in the limit as the shoreline is approached. Hibberd (1979) shows that the slope of the water surface behind the bore becomes unbounded. Ho & Meyer (1962) give more details of this shoreline 'singularity'. The motion of the shoreline after a bore has reached an otherwise still shoreline is shown by Shen & Meyer (1963) to be the same as a particle moving freely under gravity; that is, the shoreline particle is insensitive to other water motions. Thus if $u_s(t)$ is the shoreline velocity then in dimensionless variables

$$\frac{du_s(t)}{dt} = -1 \tag{12}$$

and hence

$$u_s(t) = u_0 t - \tfrac{1}{2}t^2, \tag{13}$$

where u_0 is the initial shoreline velocity. In this case the characteristics of the two families which touch the shoreline are both coincident with the shoreline. Also the depth of water near the shoreline is given by

$$h(x, t) = (x - x_s)^2/9t^2, \tag{14}$$

where x_s is the position of the shoreline, so that the water surface is tangential to the shore at the shoreline. Within a characteristic formulation the result (13) is shown to be limited in time by the appearance of a 'limit line'. This is interpreted physically as the generation of a bore within the back-wash of the motion.

Standing-wave solutions of Carrier & Greenspan differ in their behaviour near the shoreline from that as predicted above. Water depth near the shoreline in this case has the form $h(x, t) \propto (x - x_s)/t$ with the shoreline point experiencing a variable acceleration.

For a smooth wave incident upon a beach with undisturbed water preceding the wave then the wave-front is necessarily a characteristic (Jeffrey 1964). A bore incident on a beach corresponds to a more complicated four-characteristic configuration; bore relations are necessary for a complete determination of the solution. In the run-up no characteristics meet the front of the wave, as there is no water, and hence the front cannot be represented by a bore. On the other hand the shoreline need not be a single characteristic (and hence following from (6) and (7) a coincidence of characteristics). This point is clarified by the following local analysis.

Denoting the line separating the dry bed from the wet bed parametrically by $x = x(\sigma)$, $t = t(\sigma)$ and defining subscript s to represent the limit as the shoreline is approached from within the flow. Then

$$\frac{dh_s}{d\sigma} = \left.\frac{\partial h}{\partial t}\right|_s \frac{dt}{d\sigma} + \left.\frac{\partial h}{\partial x}\right|_s \frac{dx}{d\sigma} = 0, \tag{15}$$

since water height is zero along the shoreline. The derivative of water velocity

$$\frac{du_s}{d\sigma} = \left.\frac{\partial u}{\partial t}\right|_s \frac{dt}{d\sigma} + \left.\frac{\partial u}{\partial x}\right|_s \frac{dx}{d\sigma}, \tag{16}$$

where the left-hand side is non-zero in general. Equations (4) and (5) are satisfied as the shoreline is approached and give

$$\left.\frac{\partial h}{\partial t}\right|_s + u_s \left.\frac{\partial h}{\partial x}\right|_s = 0, \tag{17}$$

and

$$\left.\frac{\partial u}{\partial t}\right|_s + u_s \left.\frac{\partial u}{\partial x}\right|_s + \left.\frac{\partial h}{\partial x}\right|_s = -1. \tag{18}$$

Equations (15) to (18) yield a series of linearly dependent equations for the partial derivatives of the dependent variables at the shoreline. The condition of zero determinant gives $(dx/dt)^2 = u_s^2$, which is satisfied identically, and the resulting relations are

$$\frac{du_s}{dt} = -1 - \left.\frac{\partial h}{\partial x}\right|_s, \tag{19}$$

and

$$\left.\frac{\partial h}{\partial t}\right|_s = -u_s \left.\frac{\partial h}{\partial x}\right|_s \tag{20}$$

governing the shoreline motion. The extra term in (19) when compared with (12) arises from a finite pressure gradient seaward of the shoreline. Bore-less solutions of Carrier & Greenspan (1958) and others satisfy this new relation. We also note that the equations (19) and (20) also admit the undisturbed equilibrium solution $u_s = 0$, $\partial h/\partial x = -1$. In the case governed by (19) and (20) in which $\partial h/\partial x \not\equiv 0$ the characteristics only touch the shoreline and are otherwise distinct from each other. The shoreline is given as an envelope of characteristics.

For the special case (12) it is easily seen that there is no influence from continuous waves approaching the shoreline. No finite depth gradient will arise spontaneously, so the shoreline will recede seawards unless another bore meets it and stops its seaward motion by a discontinuous influence. This is a simple way of interpreting the 'back-wash bore' first described by Shen & Meyer (1963). We can see from the numerical solutions that when this or another bore meets the shoreline the subsequent motion is not necessarily of the form (12), thus confirming its special character.

Further discussion of analytic solutions can be found in Hibberd (1979).

4. Numerical method of solution

Two basic types of numerical methods are usually used for integrating the shallow-water equations: (i) methods based on characteristics and (ii) finite-difference methods. Previous numerical solutions for waves approaching a beach (Keller *et al.* 1960; Freeman & Le Méhauté 1964) have used characteristic methods, but there are severe drawbacks to their use when bores are involved. Special techniques need to be included to detect bores forming in the flow since they require a separate treatment in this formulation. Furthermore in the run-up characteristics become very nearly parallel and this leads to a large uncertainty in finding their point of intersection.

Finite-difference methods have been successfully used to compute shallow-water flows with bores, see particularly Houghton & Kasahara (1968). If the equations are rewritten into an appropriate conservation form and if the finite-difference representation of the derivatives is carefully chosen, then bores may be included without any special calculation. In this case the conserved quantities are mass and momentum. Such numerical schemes were introduced by Lax & Wendroff (1960) and are discussed in the textbook of Richtmyer & Morton (1967). The finite-difference scheme used is based directly on their account and includes the possibility of an additional term to minimize numerical oscillations due to nonlinear dispersive effects introduced by the discretization. These numerical oscillations appear in the form of parasitic waves near the bore. With a single bore incident on a beach these oscillations do not unduly affect the solution away from the bore or the stability of the scheme. However in further work to be reported on periodic solutions where bores advance into thin, fast-moving back-wash from previous waves such oscillations prove unacceptable.

Using a finite-difference space and time grid of space size Δx and time step Δt and defining $\mathbf{u}_{j,n} = \mathbf{u}(j\Delta x, n\Delta t)$ the finite difference equations used are

$$\mathbf{u}_{j,n+1} = \mathbf{u}_{j,n} - \lambda\{\tfrac{1}{2}[\mathbf{F}_{j+1,n} - \mathbf{F}_{j-1,n}] + \Delta x \mathbf{G}_{j,n}\} + \tfrac{1}{2}\lambda^2\{\mathbf{g}_{j,n} - \mathbf{g}_{j-1,n} - \Delta x \mathbf{S}_{j,n}\},$$

where

$$\mathbf{u} = \begin{pmatrix} m \\ h \end{pmatrix}, \quad m = uh, \quad \lambda = \Delta t/\Delta x,$$

$$\mathbf{g}_{j,n} = \tfrac{1}{2}[\mathbf{A}(\mathbf{u}_{j+1,n}) + \mathbf{A}(\mathbf{u}_{j,n})][\mathbf{F}_{j+1,n} - \mathbf{F}_{j,n} + \tfrac{1}{2}\Delta x(\mathbf{G}_{j+1,n} + \mathbf{G}_{j,n})],$$

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{u}} = \begin{bmatrix} 2m/h & h - m^2/h^2 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} m^2/h + \tfrac{1}{2}h^2 \\ m \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} h \\ 0 \end{bmatrix}$$

and

$$\mathbf{S}_{j,n} = \Delta x \frac{\partial \mathbf{G}_{j,n}}{\partial t} = -\frac{1}{2}\begin{bmatrix} m_{j+1,n} - m_{j-1,n} \\ 0 \end{bmatrix}. \qquad (21)$$

The stability criterion is that

$$\left|\frac{\Delta t}{\Delta x}\right| < \frac{1}{|u_m| + c_m} \qquad (22)$$

where u_m is the value of the maximum absolute velocity and c_m^2 is the maximum water depth. In all computations shown $\Delta t = 0{\cdot}4\Delta x$.

At the seaward boundary analysis of the characteristics for an initially subcritical bore show that the advancing characteristics enter the region of integration whilst the receding characteristics originate from within the region. We therefore specify in our numerical solution only the incoming Riemann invariant α and determine β, the outgoing invariant from within the calculation. The value of β is found by a simple first-order scheme, given at the end of the appendix, and is then combined with the prescribed value of α to determine the dependent variables.

The shoreline boundary condition is that at the shoreline position $x_s(t)$:

$$h = 0 \quad \text{and} \quad dx_s/dt = u(x_s(t), t).$$

For small, gentle shoreline motions a simple treatment, as described by Sielecki & Wurtele (1970), is adequate. However, more complex procedures are required to handle the case of a bore meeting the shoreline and the subsequent run-up and back-wash. Details of the scheme, which has proved most successful are given in the appendix to this paper. It may be thought of as giving a predictor–corrector–smoothing procedure together with a cut-off to avoid having such small water depths that errors incurred by division become a problem. Numerous other more or less complicated schemes were tried. Schemes which carefully extrapolate to the shoreline, or correspondingly use a shoreline point, are found to have little merit with this finite-difference scheme, because the small oscillations of scale Δx associated with the representation of the bore mean that one extrapolates from relatively poor data.

5. Comparisons with non-breaking analytic solutions

It is of fundamental importance that the computational scheme is checked in its ability to reproduce analytic results obtained from the same set of differential equations. Testing its performance on the various aspects of the flow yields its limitations and shortcomings and therefore results can be considered with more confidence. Known analytic solutions involving non-breaking waves are used as a basic reference. Numerical solutions are obtained for various problems in order to test the ability of the numerical scheme developed to reproduce accurate results in describing smooth waveforms away from the shoreline, shoreline position and shoreline velocity in both run-up and back-wash modes.

Figure 3 shows details of the profile near the shoreline for a wave climbing a beach of constant slope. The initial profile is as described in parametric form by equation (3.1) in the paper of Carrier & Greenspan (1958) with the amplitude coefficient taken as $\epsilon = 0{\cdot}2$. Good agreement is found with the analytic solution of Carrier & Greenspan, with the possible exception of the two grid points closest to the shoreline. These points lag behind the moving shoreline in its initial stages of motion where the shoreline accelerates rapidly from rest. At positions where the shoreline is not accelerating so rapidly better agreement is achieved.

Comparison of the numerical scheme in the back-wash with the analytic work of Spielvogel (1976) is presented in figure 4. The initial profile was taken from the paper of Spielvogel and variables suitably scaled so that results compared with the case $R = 0{\cdot}1$, $H = 0{\cdot}05$ considered explicitly. Comparison is found almost indistinguishable throughout the 600 time steps used to calculate the profile for $t = 0{\cdot}5$. Results are displayed in the same units as Spielvogel.

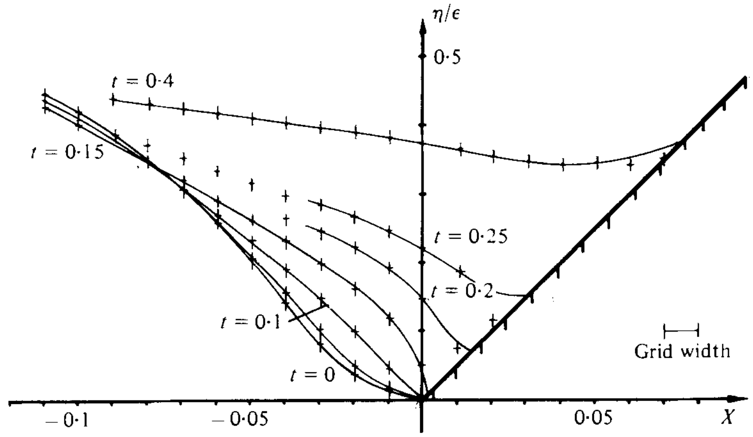

FIGURE 3. Comparison of present numerical scheme with the analytic solution of Carrier & Greenspan (1958) equation (3.1), $\epsilon = 0{\cdot}2$. + + +, present scheme; ——, exact result.

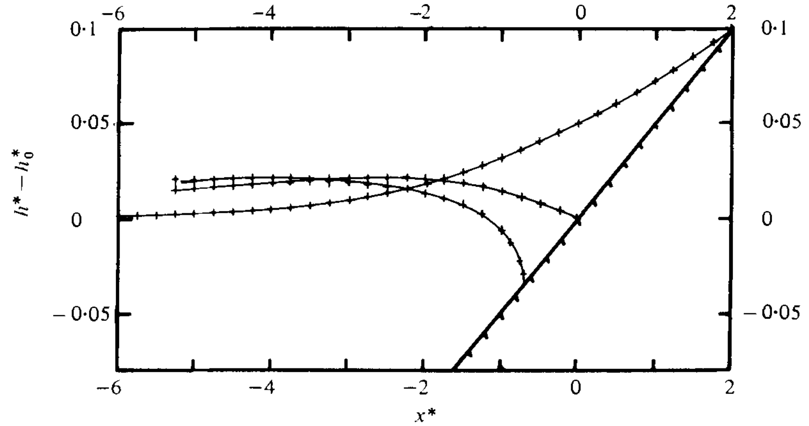

FIGURE 4. Comparison with the solution of Spielvogel (1976) for flow in the back-wash. ——, result of Spielvogel ($R = 0{\cdot}1$, $H = 0{\cdot}05$); + + +, present scheme.

The periodic solution of Carrier & Greenspan (1958) is used to test the numerical scheme over a larger number of time step integrations. A comparison with a period of the computation is shown against the analytic solution in figure 5(*a*). The largest discrepancies are found near the turning points of the shoreline velocity. However, this error is a local small-scale disturbance at the shoreline which can be seen in the complementary figure 5(*b*), showing the offshore waveform. Deviation of the wave profile from the analytic solution is only observable at these most landward grid points. If any form of smoothing on the calculated profile near the shoreline is employed substantially better accuracy is achieved. A measure of the accuracy for the whole waveform is easily implemented in that at times $t/T = 0, \frac{1}{2}\pi$, where T is the period, the water velocity is zero everywhere in the analytic solution. At the completion of the wave period the average absolute mean water velocity is found to be $2{\cdot}0 \times 10^{-4}$ which compares favourably with the expected second-order accuracy of the numerical scheme, which for this case is $O(10^{-4})$.

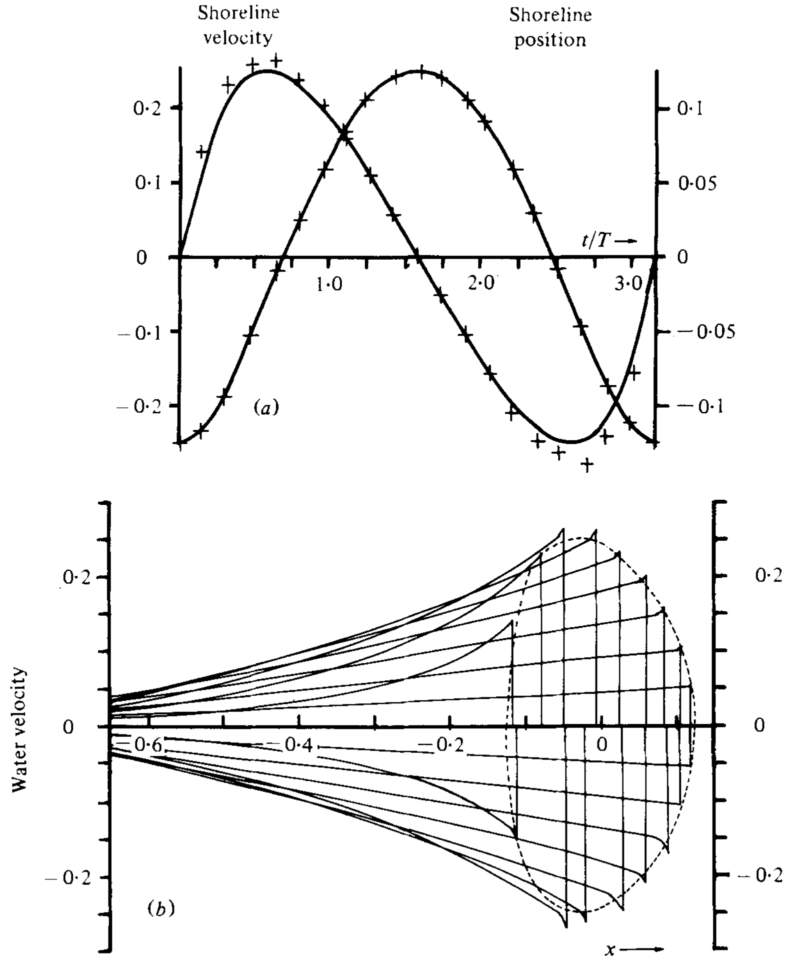

FIGURE 5. (*a*) Comparison of shoreline variables between the present scheme and the periodic solution of Carrier & Greenspan (1958). Computed points are marked + + over a wave period T. (*b*) Computed profiles of water velocity near the shoreline for solution corresponding to the exact periodic solution of Carrier & Greenspan (1958). The exact shoreline position is shown dotted.

6. Application to an incident uniform bore

Motion of a subcritical bore incident onto otherwise undisturbed water on a beach is investigated. Computation was started at the instant the bore is presumed to have reached the toe of the sloping beach and consequently the seaward boundary conditions are that $\alpha = \alpha_0 + t$. Calculation proceeded in time until all reflected waves had propagated seawards and only still water remained. A number of different bore strengths were taken and the time history of a typical example is shown in figures 6 and 7, where the initial bore height was 1·6. Figures 6 and 7 may be usefully studied in conjunction with the corresponding space–time diagrams shown as figures 8, 9 and 10. The bore height of 1·6 is chosen since it seems to be typical of bores on a beach (e.g. see Svendsen, Madsen & Hansen 1978, figure 5).

We refer first to figure 6 which shows the profile of wave elevation and velocity at time intervals of 0·4 for the offshore travel and run-up. The shoreline point in the

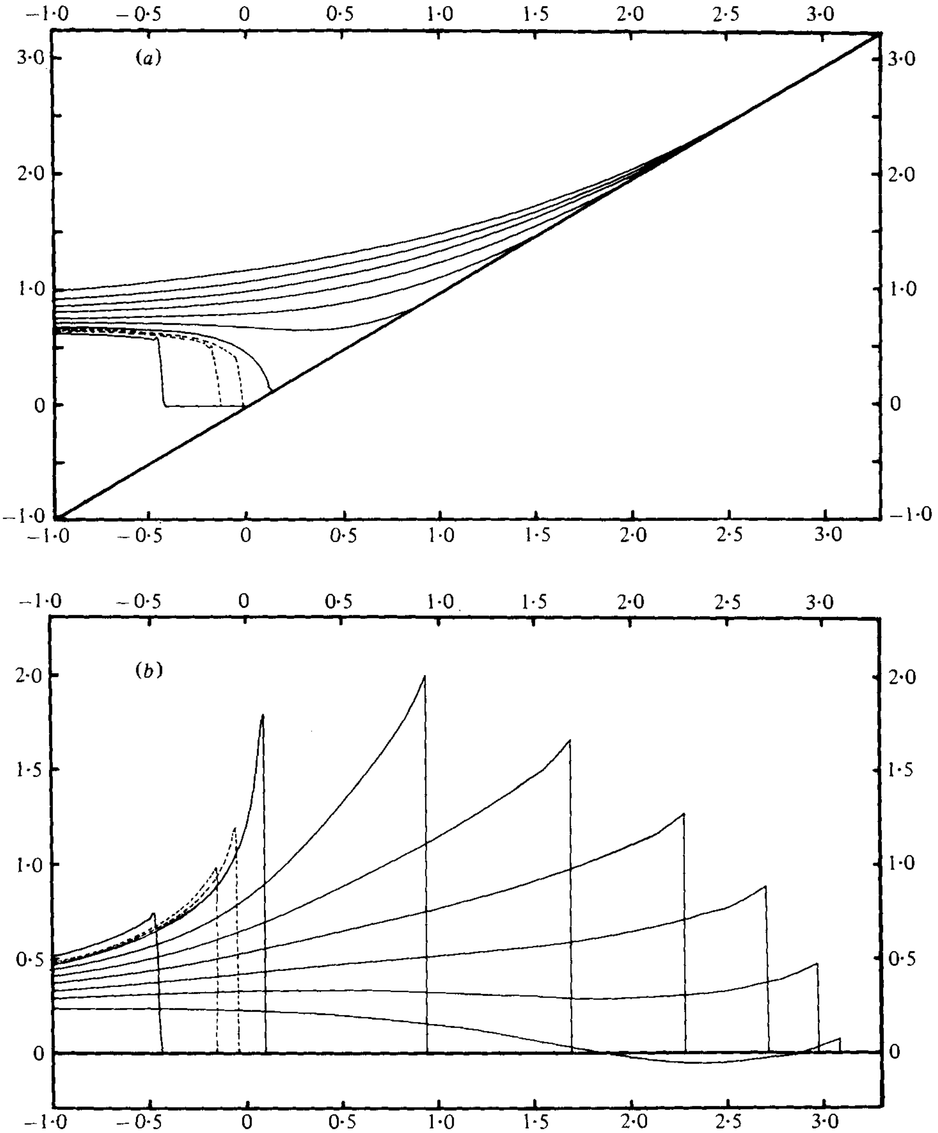

FIGURE 6. For legend see facing page.

velocity diagram is marked by a vertical line drawn to the zero velocity axis. As the incident bore encounters the sloping beach a reflected wave is formed which has the immediate effect of reducing the velocity of the water behind the incident bore. There is a build-up of slower-moving water causing a rise in the water height at the seaward boundary. The offshore behaviour closely follows the prediction of Keller *et al.* (1960) except very close to the shoreline. Large gradients in the wave elevation occur close to the shoreline together with a sharp rise in the water velocity. This rise in water velocity is found to be not so sharp or as high as the velocity predicted by the bore-tracking technique of Keller *et al.* However our numerical scheme has finite grid step which limits resolution of the details as the bore approaches the shoreline. In a real

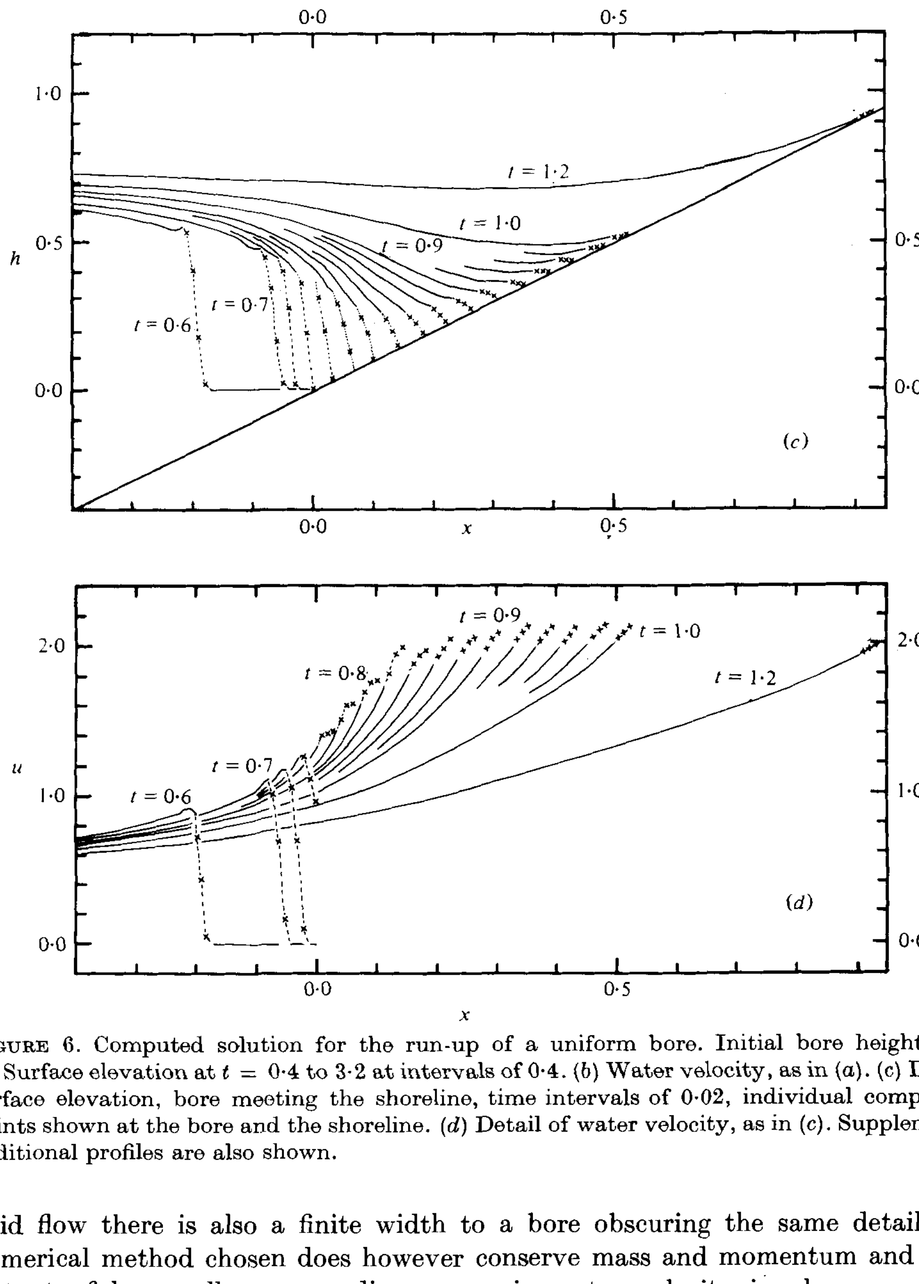

FIGURE 6. Computed solution for the run-up of a uniform bore. Initial bore height = 1·6. (*a*) Surface elevation at t = 0·4 to 3·2 at intervals of 0·4. (*b*) Water velocity, as in (*a*). (c) Detail of surface elevation, bore meeting the shoreline, time intervals of 0·02, individual computation points shown at the bore and the shoreline. (*d*) Detail of water velocity, as in (*c*). Supplementary additional profiles are also shown.

fluid flow there is also a finite width to a bore obscuring the same details. The numerical method chosen does however conserve mass and momentum and at the instant of bore collapse any discrepancy in water velocity involves very small quantities of these dependent variables.

As the bore collapses at the shoreline run-up starts. There is an initial acceleration of water velocity before the flow is fully dominated by gravity, as the analytical results suggest. That is, in the initial stages of run-up the now moving shoreline accelerates due to influence from the wave motions rearward of the shoreline. As the wave climbs the beach the gradient of wave height near the shoreline tip rapidly decreases and the run-up water depth becomes thinner. The water velocity decreases

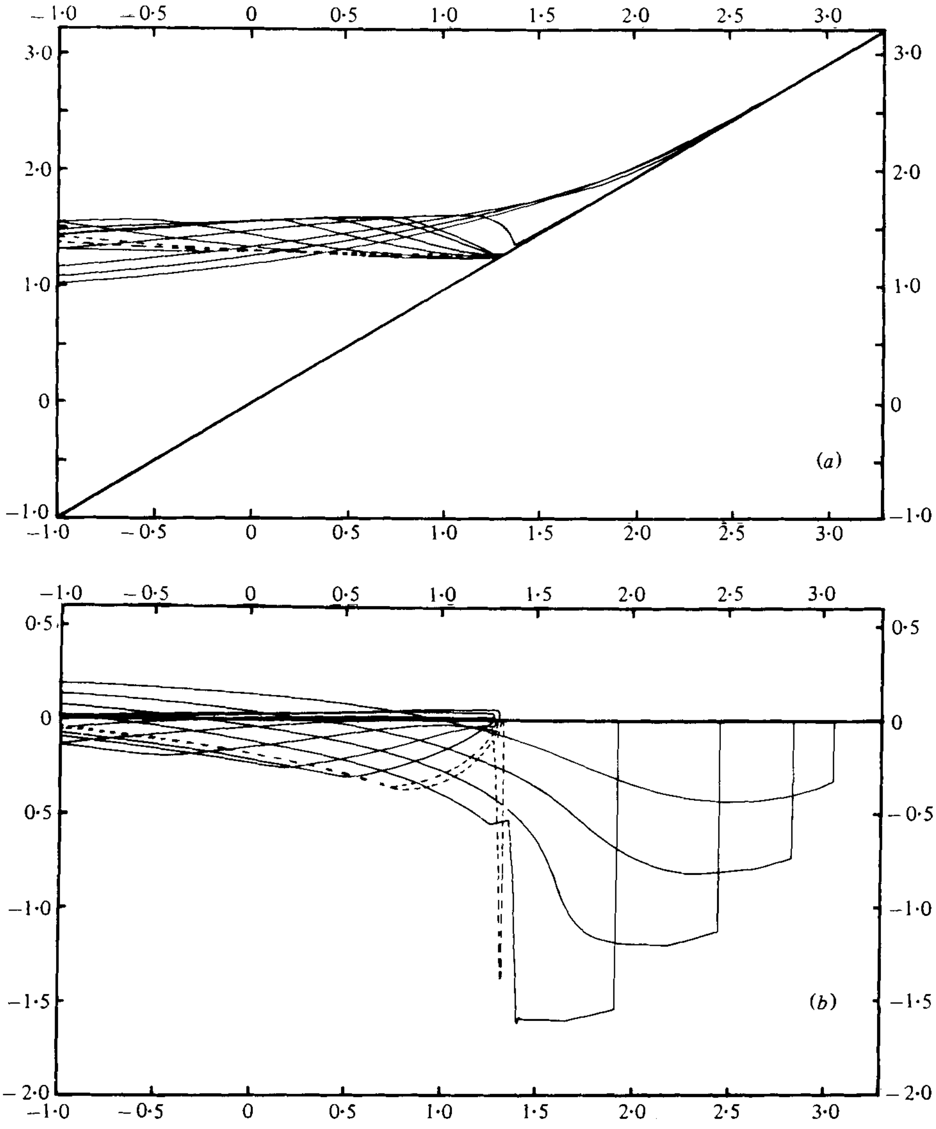

FIGURE 7. For legend see facing page.

as the extreme run-up height is approached with negative values first found at a point seaward of the run-up tip. This adds to the thinning of the run-up in its final stages. As the water velocity at the shoreline decreases to zero the water velocity at the seaward boundary also slowly decreases.

Back-wash and the formation of reflected waves are shown by the wave profiles in figure 7. Near the wave-tip, where the water is extremely thin, the back-wash velocity increases rapidly. The back-wash is retarded by slower-moving water, found where the depth is more substantial, creating an almost stationary wave steepening on the landward-facing side. This wave forms a bore lying near to the eventual new still water level and has the effect of drastically reducing the high back-wash velocities to bring the shoreline almost to rest. The back-wash bore has the characteristics of

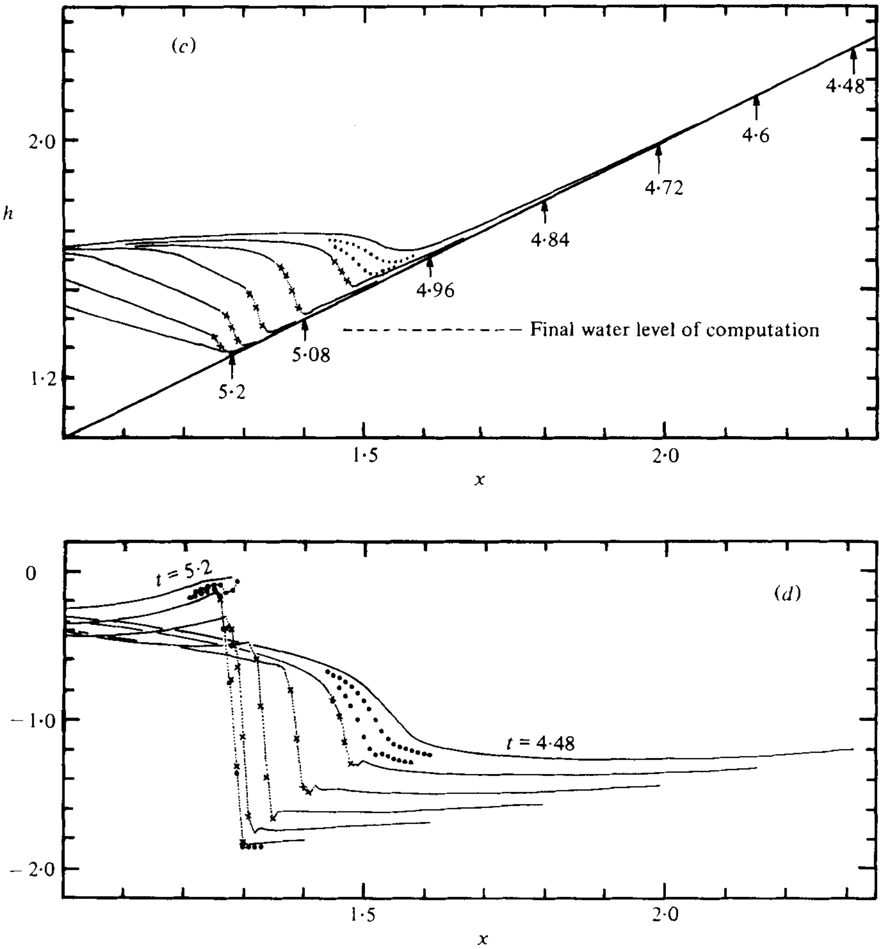

FIGURE 7. Computed solution for the backwash from a uniform bore of initial height = 1·6. (*a*) Surface elevation at t = 3·4 to 8·0 at intervals of 0·4. (*b*) Water velocity as in (*a*). (*c*) Detail of surface elevation for duration of the backwash bore, time intervals of 0·12, individual computation points are shown at the bore. Arrows indicate the shoreline position. (*d*) Detail of water velocity as in (*c*). Supplementary additional profiles are also shown.

the bore predicted by Shen & Meyer (1962) in that it is landward facing and thus not a reflected bore. As the last of the fast-moving back-wash is brought to rest the bore is depleted by water moving out through the seaward boundary. Profiles in figure 7 (*c*), (*d*) show this transformation. The extremely thin nature of the back-wash is difficult to see in the figures. The strength, h_2/h_1, of the back-wash bore does give an indication. The values of h_2/h_1 estimated from the computation at times 4·92, 5·0 and 5·12 are 50, 250 and 500 respectively. The back-wash bore has only a short life before it is converted into wave motions propagating seawards and passing outward through the seaward boundary. The shoreline makes small oscillations about the new still water level producing smooth long wave motions before eventually coming to rest. This new level provides a check on the accuracy of the numerical scheme since, as described in § 2, this final level is readily calculable. A difference between expected and computed levels was typically 0·007 of the final level attained.

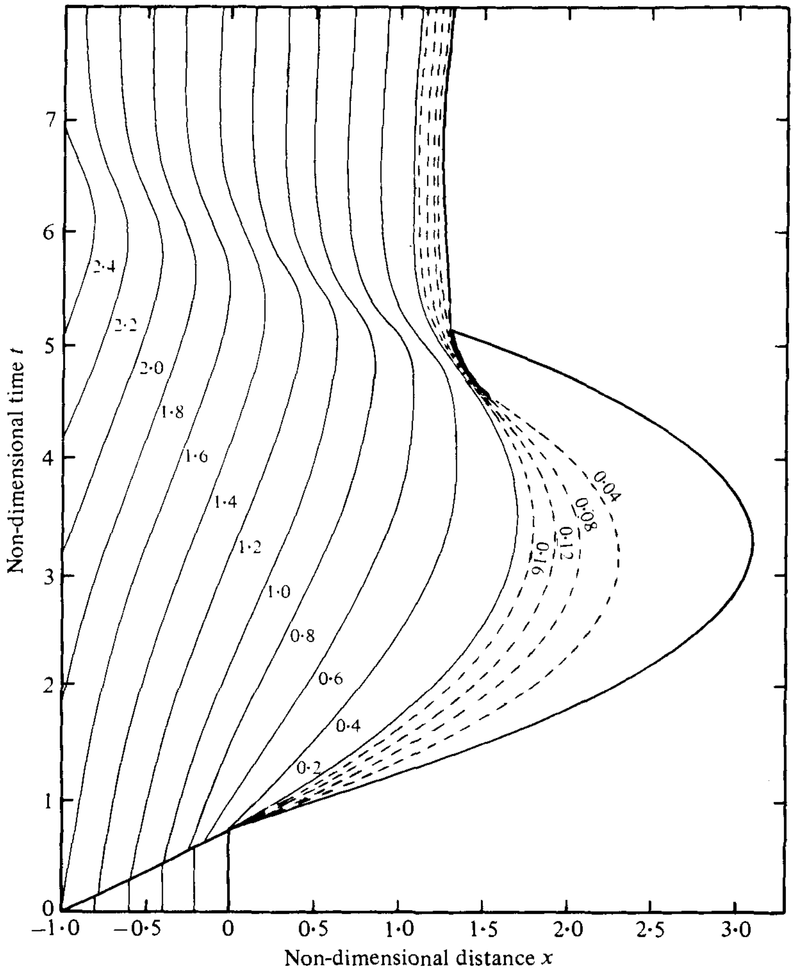

FIGURE 8. Space–time plot showing contours of water depth. Initial bore height = 1·6.

Plots of water height are drawn as contours in a space–time diagram in figures 8 and 9. Figure 8 gives the contours of water depth whilst figure 9 shows the water height above the original still water level. In the former diagram contours in the absence of motion are lines parallel to the undisturbed still water shoreline. Large gradients of wave height near the wave-front, shortly after the shoreline is set into motion, are clearly visible. Near the run-up limit the shoreline profile is approximately symmetric and parabolic since, due to very small slopes of wave height near the tip region, the shoreline motion is influenced overwhelmingly by gravity.

The same general behaviour can be seen in figure 10 showing the velocity contours. High gradients of water velocity are evident at the start of the run-up. In the back-wash velocities comparable to the uprush velocities in magnitude are produced. Again the formation of a back-wash bore and the subsequent reflected waves are clearly seen.

Figure 11 shows the pattern of advancing and receding characteristics which, since characteristics may be considered as carrying waves of infinitesimal amplitudes, depict the region of influence and domain of dependence of particular specified

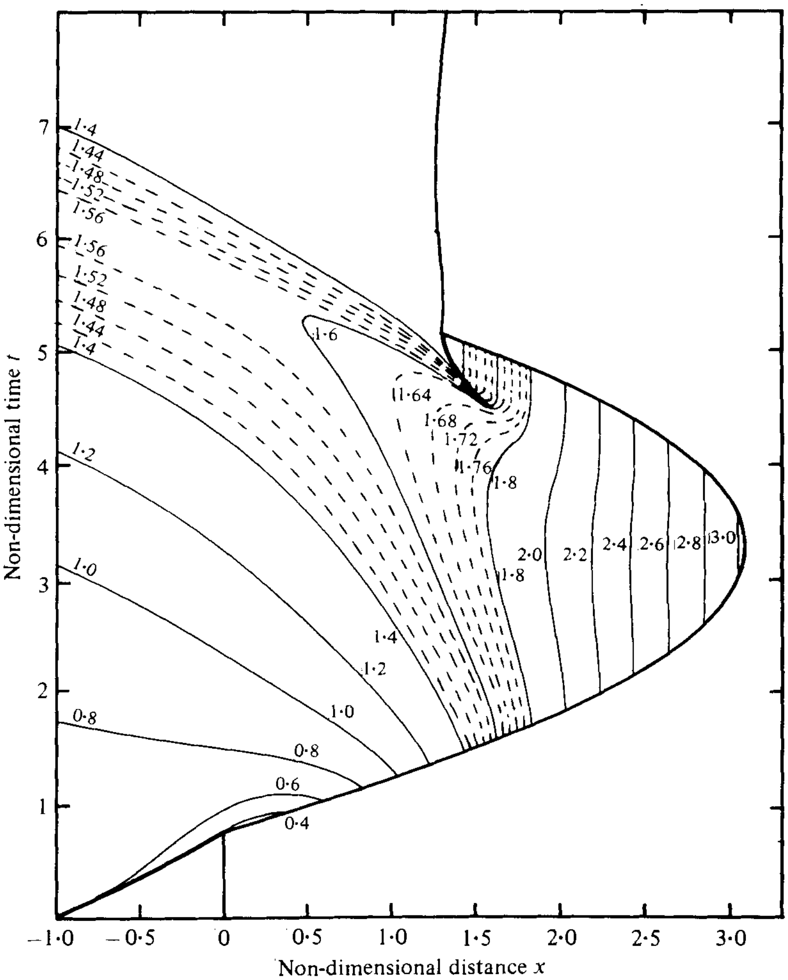

FIGURE 9. Space–time plot showing contours of water height above the original still water level. Initial bore height = 1·6.

boundary data. Advancing characteristics originate from the seaward boundary whilst receding characteristics start from within the flow region. Receding characteristics originate from either the bore or the shoreline. Near the point where the incident bore reaches the undisturbed shoreline, receding characteristics emanate into a 'fan' of characteristics reminiscent of the simple wave region associated with a horizontal dam-break solution. Even in still water such characteristics turn seawards due to the beach slope. The region of influence from the bore near the instant of bore collapse covers most of the shoreline motion. The initial shoreline motion is influenced only by a section of the seaward boundary extending over a period of only about 25 % of the total time taken for the bore to reach the shoreline. The entire run-up and back-wash, until the appearance of a back-wash bore, is controlled by data specified for only about 10 % of the total time the shoreline takes to reach that position. Of course additional data is required for a description of wave motions within the interior of the flow.

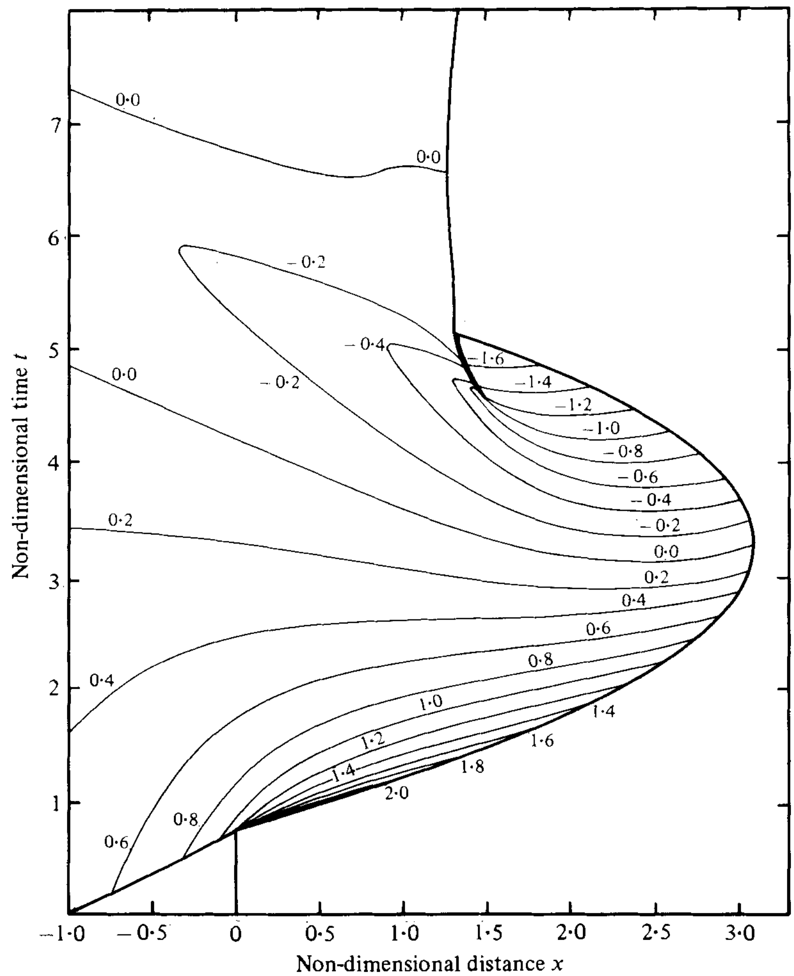

FIGURE 10. Space–time plot showing contours of water velocity. Initial bore height = 1·6.

A concentration of advancing characteristics in the back-wash show the formation of a back-wash bore and the subsequent pattern of characteristics at the gently oscillating shoreline. An envelope of advancing and receding characteristics form the shoreline after the back-wash bore intersects the receding shoreline. The form of the characteristic net near the shoreline in the run-up and back-wash modes is not identifiable due to the concentration of characteristics coupled with the finite resolution. Since water of depth less than δ was deleted, δ was set at 10^{-4} throughout the computation, a correction term to the shoreline should be estimated. An upper bound on the shoreline is obtained by taking the local solution given by Shen & Meyer (1963). With this prediction the amount to be added at the run-up limit of figures 8 to 11 is less than 1 % of the horizontal run-up distance.

7. Discussion

Figures 6 to 11 clearly depict the succession of stages involved when a single uniform bore is incident onto a sloping beach. The offshore bore motion is transformed at the

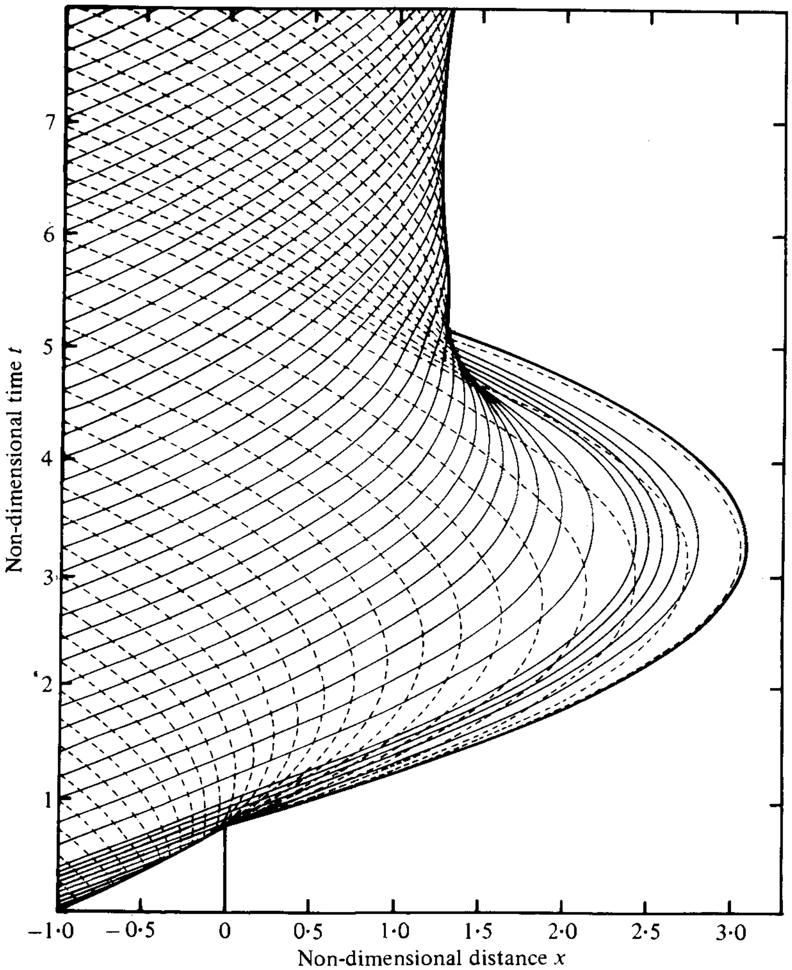

FIGURE 11. Space–time plot showing net of advancing (——) and receding (- - -) characteristics. Initial bore height = 1·6.

initially undisturbed shoreline into a wave that surges up the beach and consequently followed by the back-wash. In the latter flow regimes the shoreline acceleration is very nearly the component of gravity down the beach slope. Gradients of water height at the wave-tip are small. The appearance of a second landward-facing bore, which is a known feature in long period surf, brings this receding shoreline to rest. The ensuing motion consists of the shoreline gently oscillating to rest whilst the interior wave motions propagate seawards.

In figure 12 a comparison of the run-up heights obtained from a series of computations is made with the analytic approximations as given by Meyer & Taylor (1972). Also marked are best-fit curves through experimentally determined run-up heights for a uniform bore as determined by Miller (1968). The results quoted by Meyer & Taylor are a combination of taking the motion of the shoreline as dominated by gravity and to take as the initial shoreline velocity the value determined from the approximation to offshore bore travel suggested by Whitham (1958). Agreement is

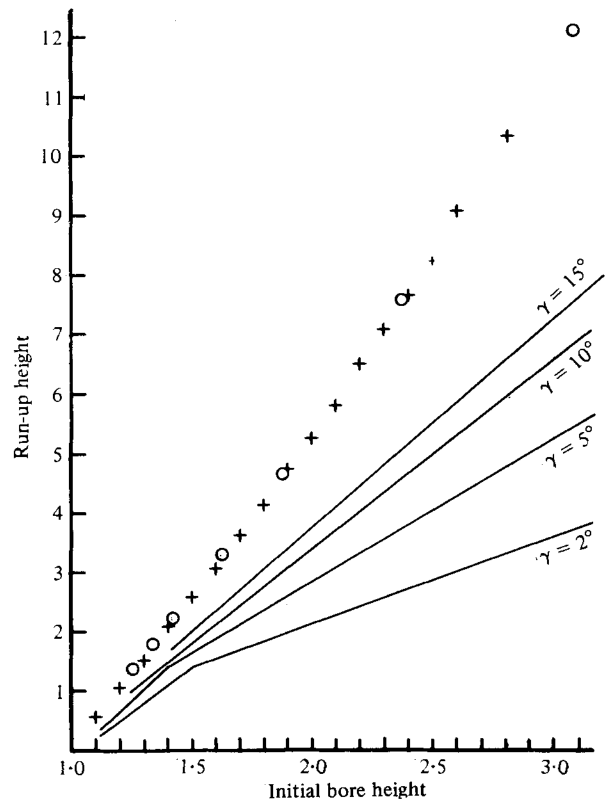

FIGURE 12. Comparison of results for run-up heights.

found to be extremely close; this is because in the majority of the run-up the gradient of water depth at the wave-tip is small. Also for an initially uniform bore the Whitham approximation also produces a remarkable accuracy (Hibberd 1979) in determining the initial shoreline velocity. The experimentally generated curves lie somewhat below the theoretical curves. Although Miller used a 'smooth' slope for his experiments friction effects are apparent. In the absence of any friction the curves for each beach angle should be almost identical once scaled in the manner of this paper. Friction effects are more evident on the smaller beach angles where the run-up lengths are greater. It is interesting to note that the curves for the larger beach slopes of 10° and 15° are much closer than for the smaller slope angles since the friction effects are less pronounced due to reduced run-up lengths. The effect of friction is to reduce the run-up height; thus from figure 12 it is clear that addition of a friction term could well bring the numerical results in line with the experimental results.

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Appendix

Treatment of boundary points. Near the moving shoreline and at the seaward boundary the numerical scheme (21) needs to be supplemented by boundary values. At the shoreline the revised scheme has to predict whether a new grid point must be provided for the run-up or if in the back-wash whether a grid point is no longer covered by water and therefore must cease to be included in further computations. If a new grid point is introduced the values of the variables at that point must be calculated. With a new grid point introduced into the computation known values at grid points on the lower time level are insufficient to use the central-difference scheme (21) directly. Near the shoreline errors can be magnified due to division by small values of water depth in order to determine the water velocity. To avoid these difficulties, water velocity is used near the shoreline as a dependent variable and a lower bound, δ say (usually taken as 10^{-4}), on the water depth is set, below which the water depth is considered zero. In general the position of the shoreline does not coincide with a grid point and it is therefore the last underwater point, denoted by subscript s, which requires special attention. At each time step the shoreline does not cross more than a single grid interval if the stability criterion (22) is satisfied.

The method of calculation of values at grid points near the shoreline differs according to the direction of motion at the shoreline.

(*a*) *Run-up*

Assuming all values are known at time level $n\Delta t$ we are required to solve explicitly for $h_{s,n+1}$, $h_{s+1,n+1}$, $u_{s,n+1}$ and $u_{s+1,n+1}$. The following procedure is followed.

(i) Obtain values at grid index $(s+1, n)$ by a linear extrapolation from values at grid points (s, n) and $(s-1, n)$.

(ii) Use the Lax–Wendroff scheme to obtain values $(s, n+1)$.

(iii) Calculate provisional values at grid point $(s+1, n+1)$, denoted by $u^*_{s+1,n+1}$ and $h^*_{s+1,n+1}$, by linear extrapolation from values at $(s, n+1)$ and $(s-1, n+1)$. What then follows then depends on whether $h^*_{s+1,n+1} > \delta$ (case I) and final values at $(s+1, n+1)$ are calculated, or $h^*_{s+1,n+1} \leqslant \delta$ (case II) in which case the shoreline is deemed not to have reached the grid position $(s+1, n+1)$ and corrected values at $(s, n+1)$ are calculated.

Case I.

(iv) The Lax–Wendroff scheme is used to provide provisional values at $(s, n+2)$ using grid points $(s-1, n+1)$, $(s, n+1)$ and $(s+1, n+1)$.

(v) Use a central difference scheme of mass and momentum equations centred at $(s, n+1)$ to give

$$m_{s+1,n+1} = m_{s-1,n+1} - \frac{\Delta x}{\Delta t}(h^*_{s,n+2} - h_{s,n}) \tag{A 1}$$

and

$$u_{s+1,n+1} = u_{s-1,n+1} - \left[\frac{\Delta x}{\Delta t}(u^*_{s,n+2} - u_{s,n}) + h^*_{s+1,n+1} - h_{s-1,n+1} + 2\Delta x\right] \Big/ u_{s,n+1}. \tag{A 2}$$

If $|u_{s+1,n+1}| < \delta$ then (v) is replaced by

$$h_{s+1,n+1} = h^*_{s+1,n}$$

and, using the relation (19) for the water front,

$$u_{s+1,n+1} = u_{s+1,n} + \frac{\Delta t}{\Delta x}(h_{s,n} - h_{s+1,n} - \Delta x). \tag{A 3}$$

In (A 3) the relation valid at the shoreline tip is applied at the most shoreward grid point; in general such an error is $O(\Delta x)$ but due to low gradients of velocity where this method is applied the accuracy proves sufficient.

Case II. The grid point $(s+1,\ n+1)$ is not, as yet, included in the computation and instead a correction scheme for values at the grid point $(s,\ n+1)$ is implemented. The procedure follows case I with s replaced throughout by $s-1$.

The first steps (i) and (ii) ensure smooth gradients at the shoreline so as not to cause small-scale disturbances, which in the absence of smoothing, propagate into the flow, contaminating the results. Division of the following procedure into two separate cases has the advantage of always correcting the values closest to the shoreline at each step. Where the shoreline is fast moving most attention is paid to the new grid points created. Where the shoreline is moving slowly then both water depth and velocity are small in the vicinity of the shoreline and are more susceptible to small scale disturbances; the correction helps to remedy this.

(*b*) *Back-wash*

No difficulties are encountered in calculating shoreline points in the back-wash. No new grid points are added to the computation. We need only to calculate values at grid point $(s,\ n+1)$ with the possibility of the shoreline having receded beyond that point. The method is as follows:

(i) Calculate values at $(s+1,\ n)$ by linear extrapolation from values at $(s-1,\ n)$ and $(s,\ n)$.

(ii) Use the Lax–Wendroff scheme to calculate values at $(s,\ n+1)$. If the value of the water depth $h_{s,n+1} < \delta$ then that point is no longer active in the computation of the back-wash.

The seaward boundary condition is taken as a given by specifying a value for the positive Riemann invariant. For an initially uniform bore the appropriate condition is given by (11). In order to obtain values of the dependent variables on the seaward boundary the value of the negative Riemann invariant β is to be given. For subcritical motion the value for β is determined from within the region of integration. The differential form satisfied by β is determined from (8) and (9) as

$$\partial\beta/\partial t + (u-c)\ \partial\beta/\partial x = 0.$$

On performing a one-sided differencing of the above equation we obtain a value for $\beta_{sb,n+1}$, the value at the seaward boundary at time level $(n+1)\Delta t$ by

$$\beta_{sb,n+1} = \beta_{sb,n} - \frac{\Delta t}{\Delta x}[u_{sb,n} - c_{sb,n}][\beta_{sb+1,n} - \beta_{sb,n}]. \tag{A 4}$$

Values of the dependent variables are given from using (6) and (7).

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